Optimization of Interplanetary Orbit Transfers by Dynamic Programing

Cornelius T. Leondes*
University of California, Los Angeles, Calif.

AND

Frederick T. Smith†
Singer-Librascope, Glendale, Calif.

A low-thrust interplanetary orbit transfer process is approximated by a piecewise linear process. The process includes the effects of perturbing gravitational forces from Earth, Mars, and Jupiter. The process is optimized by the method of dynamic programming. Convergence of the iterative computational process to an approximate solution is investigated by contraction operator theory. Convergence of the approximate solution to the actual solution of the differential equations and constraints defining the orbit transfer as the number of decision stages approaches infinity follows from well-known existence theorems in differential equation theory. Numerical results are presented to show the effect of the number of decision stages on convergence in orbit transfer space. In addition, the effect of the number of decision stages on the convergence of Δa, the error component for the semimajor axis, is determined in function space. Some effects of integration errors on the capture conditions at the termination of the orbit transfer are also presented.

Introduction

THE orbit transfer problems to be considered in this paper involve the modification of the motion of a space vehicle moving in a conservative force field. This motion is to be changed from that defined by some known initial trajectory to that defined by some desired terminal trajectory in accordance with some specified performance criteria. The objective of an orbit transfer may be to correct the orbit of a 24-hr communication satellite to one more nearly circular, more nearly equatorial, and with a more precise period. Other objectives might be the transfer of a space vehicle from the Earth's orbit to the orbit of Mars by the use of a low-thrust propulsion system, and the rendezvous of a ferry vehicle with an Earth satellite.

The motion of the vehicle can be represented by the time behavior of the variables forming the set of solutions to the differential equations of its motion. The choice of a suitable coordinate system depends upon the particular circumstances under which the motion takes place. Usually, the motion of the space vehicle also must satisfy a set of performance requirements that can be represented mathematically by a functional. When system performance requirements are satisfied by the vehicle's motion, this functional assumes a minimum or maximum value. The functional often consists of a measure of the terminal errors in the variables describing the vehicle's motion and a measure of the energy used during the orbit transfer.

Equations of Motion

When the attraction of the sun is the only force acting on a space vehicle it will travel along a conic section, say an ellipse, with one of its foci located at the sun. It is well known from celestial mechanics that an elliptic orbit can be defined by a set of six parameters. These parameters define the size and shape of the orbit, its orientation in space, and the time at which the vehicle was at a particular point on the orbit. There are many possible sets of orbital parameters which have useful properties for special circumstances. Any particular set of these parameters will be denoted by p_i, $i = 1,2, \ldots, 6$ and represented as an orbital parameter vector

$$\mathbf{p} = (p_1, \ldots, p_6)^T \tag{1}$$

In an inverse square force field vector $\mathbf{p}$ is constant. However, when rocket thrust or perturbing forces from natural causes act on the space vehicle the values of the orbital parameters change with time. It is shown in most books on celestial mechanics that the time behavior of the orbital parameters caused by perturbing forces is defined, to a first-order approximation, by six ordinary first-order differential equations which can be expressed in vector matrix term as

$$d\mathbf{p}/dt = A(\mathbf{p};t)\,\mathbf{f}_w(t) \tag{2}$$

where $f_w(t)$ represents the time behavior of the perturbing forces caused by rocket thrust and natural causes.

The vector-matrix matrix differential equation defining the time behavior of the orbital parameters can be formally integrated to yield

$$\mathbf{p}(t_0 + \tau) = \mathbf{p}(t_0) + \int_{t_0}^{t_0+\tau} A(\mathbf{p};t)\,\mathbf{f}_w(t)\,dt \tag{3}$$

The integral can be approximated by a midpoint quadrature routine[1]

$$\int_{t_0}^{t_0+\tau} A(\mathbf{p};t)\,\mathbf{f}_w(t)\,dt \approx \sum_{i=1}^{N} A(\mathbf{m}_i;n_i)\mathbf{f}_w(n_i)\,\Delta t \tag{4}$$

where

$$\mathbf{m}_i = (\mathbf{p}_{i+1} + \mathbf{p}_i)/2, \quad n_i = (t_{i+1} + t_i)/2 \tag{5}$$

$$\Delta t = t_{i+1} - t_i, \quad \tau = N\Delta t \tag{6}$$

Defining

$$\Delta t\mathbf{f}_w(n_i) = \Delta\mathbf{W}_i \tag{7}$$

Received January 8, 1969; revision received December 31, 1970. The authors express their appreciation to The Rand Corporation for permission to reproduce certain figures and material in Ref. 5 for use in this paper.

* Professor of Engineering. Member AIAA.

† Manager, Systems Analysis. Member AIAA.

we can write

$$\mathbf{p}_{i+1} = \mathbf{p}_i + A_i\Delta\mathbf{W}_i \tag{8}$$

where $\mathbf{p}_i = \mathbf{p}(t_i)$, and $A_i = A(\mathbf{m}_i;n_i)$.

If the desired terminal orbit at $t_0 + \tau$ is denoted by the 6×1 vector $\mathbf{p}_T$, then substracting both sides of the difference equation for $\mathbf{p}_{i+1}$ from vector $\mathbf{p}_T$ yields

$$\Delta\mathbf{p}_{i+1} = \Delta\mathbf{p}_i - A_i\Delta\mathbf{W}_i, \; i = 0,1, \ldots , N - 1 \tag{9}$$

where

$$\Delta\mathbf{p}_i = \mathbf{p}_T - \mathbf{p}_i, \; \Delta\mathbf{W}_i = \Delta\mathbf{U}_i + \Delta\mathbf{V}_i \tag{10}$$

where $\Delta\mathbf{U}_i$ = velocity vector increment due to natural perturbing forces acting during the $(i + 1)$th interval, and $\Delta\mathbf{V}_i$ = velocity vector increment due to rocket thrust acting during the $(i + 1)$th interval. This equation gives a piecewise, approximate solution for the orbital parameter correction vector $\Delta\mathbf{p}_{i+1}$ during the orbit transfer process.

System Performance Indexes

In an orbit transfer process, the objective is to change the motion of the space vehicle or satellite from some initial orbit to some desired terminal orbit. Therefore, it will be assumed that some measure of the terminal errors in the state variables is to be minimized. The mass of rocket fuel consumed during the orbit transfer is another important factor. Thus, a fuel constraint term will also be included in the system performance index.

At the termination of the orbit transfer the orbit parameter correction vector becomes

$$\Delta\mathbf{p}(t_0 + \tau) = \mathbf{p}_T - \mathbf{p}(t_0 + \tau) \tag{11}$$

and its components are the negative values of the terminal errors. Since we are to consider an N-stage decision process, we can denote $\Delta\mathbf{p}(t_0 + \tau)$ by $\Delta\mathbf{p}_N$ and express the sum of the weighted squares of the terminal errors as the quadratic form

$$\Delta\mathbf{p}_N{}^T Q \Delta\mathbf{p}_N \tag{12}$$

where the 6×6 matrix Q_N is a unit matrix if the errors are all given.

The form of the fuel constraint depends on the space vehicle's propulsion system. Consider a low-acceleration orbit transfer using a nuclear propulsion system. The reactor thermal power P_r is related to the exhaust jet kinetic energy by the expression[2]

$$P_r = -(^1/_2\eta_j)\,(dm/dt)\, c^2(t) \tag{13}$$

where

$$\|\mathbf{a}\| = -[1/m(t)]\,(dm/dt)c^2(t) \tag{14}$$

If we eliminate $c(t)$ from these two equations and integrate we obtain

$$\frac{1}{m(t_0 + \tau)} = \frac{1}{m(t_0)} + \frac{1}{2\eta_j P_r}\int_{t_0}^{t_0+\tau} \|\mathbf{a}(t)\|^2\, dt \tag{15}$$

Minimizing the fuel consumed is equivalent to maximizing the mass of the space vehicle at the termination of the orbit transfer. This is accomplished by making P_r as large as possible and choosing vector $\mathbf{a}(t)$ to minimize the integral during the orbit transfer.

We shall be considering a discrete decision process. Therefore, the integral in the previous expression must be approximated by a summation before it can be incorporated into a performance index. If we divide the time interval $t_0 \leqq t \leqq t_0 + \tau$ into N equal subintervals Δt, one for each decision stage, then the integral can be approximated by

$$\int_{t_0}^{t_0+\tau} \|\mathbf{a}(t)\|^2\, dt \approx \sum_{k=0}^{N-1} \|\mathbf{a}(t_k)\|^2 \cdot \Delta t \tag{16}$$

where $\Delta t = t_{k+1} - t_k$, $k = 0, \ldots , N - 1$. Since $\mathbf{a}(t_k)$ is a constant vector over each subinterval we can write

$$\|\mathbf{a}(t_k)\| = \|\Delta\mathbf{V}_k\|/\Delta t \tag{17}$$

where $\|\Delta\mathbf{V}_k\|$ is the Euclidean norm of the velocity vector increment caused by rocket thrust acquired by the space vehicle over the $(k + 1)$th subinterval. Therefore, the integrals can be represented approximately by

$$\int_{t_0}^{t_0+\tau} \|\mathbf{a}(t)\|^2\, dt \approx \frac{1}{\Delta t}\sum_{k=0}^{N-1} \Delta\mathbf{V}_k{}^T\Delta\mathbf{V}_k \tag{18}$$

If we combine the quadratic fuel constraint summation Eq. (18), with the quadratic terminal error measure Eq. (12), we obtain the performance index for the N-stage decision process representing the orbit transfer

$$J_N = \Delta\mathbf{p}_N{}^T Q_N \Delta\mathbf{p}_N + \lambda \sum_{k=0}^{N-1} \Delta\mathbf{V}_k{}^T\Delta\mathbf{V}_k \tag{19}$$

The constant factor $1/\Delta t$ has been absorbed into the weighting constant λ. This performance index is minimized for a given value of λ by choosing the optimal set of $\Delta\mathbf{V}_k$ vectors. Adjusting the value of λ weights the minimization of terminal errors against the minimization of the fuel constraint.

Optimization by Dynamic Programing

When the time behavior of a set of two-body orbital parameters is used to define the space vehicle's motion, it is convenient to define the orbit transfer process as a discrete decision process. The optimal policy for such a process consists of a set of N 3×1 vectors $\Delta\mathbf{V}_k$, $k = 0,1,\ldots, N - 1$. Each $\Delta\mathbf{V}_k$ represents the vector increment in the space vehicle's velocity vector during the $(k + 1)$th stage of the orbit transfer process caused by rocket thrust.

We begin with a definition of the minimized performance index or cost function $f_N(\Delta\mathbf{p}_0)$ = the cost of an orbit transfer process of N stages, when the initial stage is $\Delta\mathbf{p}_0$, and an optimal decision process is used. The principle of optimality for an orbit transfer process represented as a discrete decision process can be expressed as follows:[3,4] An optimal sequence of incremental velocity vectors $\Delta\mathbf{V}_0,\Delta\mathbf{V}_1, \ldots , \Delta\mathbf{V}_{N-1}$ has the property that whatever the initial state $\Delta\mathbf{p}_0$ may be and whatever choice is made for ΔV_0, the remaining sequence $\Delta\mathbf{V}_1, \Delta\mathbf{V}_2, \ldots , \Delta\mathbf{V}_{N-1}$ must constitute an optimal sequence with regard to the state $\Delta\mathbf{p}_1$ resulting from the choice of ΔV_0.

Recognizing that the terminal error term only contributes to the value of the performance index for the last stage of the process we have

$$f_N(\Delta\mathbf{p}_0) = \operatorname*{Min}_{\Delta V_0}[\lambda\Delta\mathbf{V}_0{}^T\Delta\mathbf{V}_0 + f_{N-1}(\Delta\mathbf{p}_1)] \tag{20}$$

The first term inside the brackets represents the cost of the process during the initial stage while $f_{N-1}(\Delta\mathbf{p}_1)$ represents the cost of the process during the remaining $N - 1$ stages which, by the principle of optimality, must be a minimum. This functional relationship can be used along with the difference equation for the orbital parameter correction vector derived earlier to obtain the following set of recurrence equations for an N-stage decision process.[3,5]

$$\begin{aligned} Q_k &= Q_{k+1} - Q_{k+1}A_k\,(A_k{}^T Q_{k+1} A_k + \lambda I)^{-1}\, A_k{}^T Q_{k+1} \\ Q_N &= \mathrm{I} \end{aligned} \tag{21}$$

$$\Delta\mathbf{V}_k = (A_k{}^T Q_{k+1} A_k + \lambda I)^{-1}\, A_k{}^T Q_{k+1} \times \left(\Delta\mathbf{p}_k - \sum_{i=k}^{N-1} A_i \Delta\mathbf{U}_i\right) \tag{22}$$

$$\Delta\mathbf{p}_k = \Delta\mathbf{p}_{k-1} - A_{k-1}\Delta\mathbf{U}_{k-1} - A_{k-1}\Delta\mathbf{V}_{k-1} \tag{23}$$

$$f_N(\Delta \mathbf{p}_0) = \left(\Delta \mathbf{p}_0 - \sum_{i=0}^{N-1} A_i \Delta \mathbf{U}_i\right)^T Q_0 \times \left(\Delta \mathbf{p}_0 - \sum_{i=0}^{N-1} A_i \Delta \mathbf{U}_i\right), k = 0, 1, \ldots, N-1 \quad (24)$$

Iteration Process

In the recurrence equations derived previously, the elements of matrix A change their values from stage to stage as the values of the orbital parameters change during the orbit transfer process. Since the time behavior of the orbital parameters during the orbit transfer process is not known beforehand, an initial estimate of their behavior must be made and an iteration process used.

The iteration process is started by assuming that the orbital parameters change by an equal amount during each stage of the process, i.e.,

$$\Delta \mathbf{p}_{k+1} = \Delta \mathbf{p}_k - (\Delta \mathbf{p}_0/N)$$

A tentative set of orbital parameter vectors p_k, $k = 1, 2, \ldots, N-1$, is computed on the basis of this assumption, and the A matrices are evaluated. Based on this tentative set of matrices the following equations can be alternatively evaluated starting with Δp_0 until Δp_{N-1} is obtained

$$\Delta \mathbf{V}_k = (A_k^T Q_{k+1} A_k + \lambda I)^{-1} A_k^T Q_{k+1} \left(\Delta \mathbf{p}_k - \sum_{i=k}^{N-1} A_i \Delta \mathbf{U}_i\right)$$

$$\Delta \mathbf{p}_k = \Delta \mathbf{p}_{k-1} - A_{k-1} \Delta \mathbf{U}_{k-1} - A_{k-1} \Delta \mathbf{V}_{k-1}$$

$$\Delta \mathbf{U}_{k-1} = \mathbf{f}_u(t_{k-1}) \, \Delta t$$

where $\mathbf{f}_u(t_k)$ is the perturbing force per unit mass due to natural causes. This force usually depends on the position of the space vehicle, and thus the time behavior of $\mathbf{f}_u(t)$ depends on the time behavior of the orbital parameters during the orbit transfer process and on the motions of the perturbing bodies.

The set of vectors $\Delta \mathbf{p}_k$, $k = 0, 1, \ldots, N-1$, so obtained is used to compute a new set of orbital parameter vectors $\mathbf{p}_k$, $k = 0, 1, \ldots, N-1$. This set is used in turn to compute a new set of A and Q matrices, and the process is repeated until satisfactory convergence is obtained. A set of control matrices

$$\Gamma_k = (A_k^T Q_{k+1} A_k + \lambda I)^{-1} A_k^T Q_{k+1}$$

is formed for use in the optimal control process. The set of incremental velocity vectors due to natural perturbing forces, $\Delta \mathbf{U}_k$, $k = 0, 1, \ldots, N-1$ has been evaluated along the optimal trajectory, as has been the final set of A matrices.

In the next section the previous equations will be put in a better form for computing the iteration process, and conditions for convergence will be determined.

Convergence of the Iteration Process

The matrix recurrence equations derived in the preceding section must be used in conjunction with an iteration process when an actual orbit transfer process is to be computed. This is caused by the fact that the elements of the matrices in the recurrence relations are functions of the orbital parameters that change their values from stage to stage of the process (these sets of values are unknown initially).

The convergence process can be divided into two parts: convergence of the iteration process to be approximate solution and the convergence of the approximate solution to the actual solution of the differential equations and constraints defining the orbit transfer. The first part has been investigated by the theory of contraction operators.[6] The second part follows from well-known existence theorems in differential equation theory.[7]

If we substitute for $\Delta \mathbf{V}_k$ in the equation for $\Delta \mathbf{p}_{k+1}$ we obtain

$$\begin{aligned}\Delta \mathbf{p}_{k+1} &= \Delta \mathbf{p}_k - A_k \Delta \mathbf{U}_k - A_k (A_k^T Q_{k+1} A_k + \lambda I)^{-1} A_k^T Q_{k+1} \Delta \mathbf{p}_k + A_k (A_k^T Q_{k+1} A_k + \lambda I)^{-1} A_k^T Q_{k+1} \sum_{i=k}^{N-1} A_i \Delta \mathbf{U}_i = \\ &[I - A_k (A_k^T Q_{k+1} A_k + \lambda I)^{-1} A_k^T Q_{k+1}] \Delta \mathbf{p}_k - [I - A_k (A_k^T Q_{k+1} A_k + \lambda I)^{-1} A_k^T Q_{k+1}] \sum_{i=k}^{N-1} A_i \Delta \mathbf{U}_i + \sum_{i=k}^{N-1} A_i \Delta \mathbf{U}_i - A_k \Delta \mathbf{U}_k\end{aligned}$$

Define the perturbation vector $\mathbf{r}_k$ by

$$\mathbf{r}_k = \sum_{i=k}^{N-1} A_i \Delta \mathbf{U}_i$$

Then

$$A_k \Delta \mathbf{U}_k = \mathbf{r}_k - \mathbf{r}_{k+1}$$

From the recurrence relation for Q_k we also have

$$Q_{k+1}^{-1} Q_k = I - A_k (A_k^T Q_{k+1} A_k + \lambda I)^{-1} A_k^T Q_{k+1}$$

The equation for $\Delta \mathbf{p}_{k+1}$ becomes

$$\Delta \mathbf{p}_{k+1} = Q_{k+1}^{-1} Q_k \Delta \mathbf{p}_k - Q_{k+1}^{-1} Q_k \mathbf{r}_k + \mathbf{r}_{k+1}$$

or

$$(\Delta \mathbf{p}_{k+1} - \mathbf{r}_{k+1}) = Q_{k+1}^{-1} Q_k (\Delta \mathbf{p}_k - \mathbf{r}_k)$$

This equation defines the behavior of the orbital parameter correction vector along the approximation of the optimal orbit transfer trajectory in orbital parameter space. It should be noted that matrix Q_k is a function of the components of orbital parameter vector $\mathbf{p}(t_k)$.

If we substitute the corresponding relation for $(\Delta \mathbf{p}_k - \mathbf{r}_k)$ we obtain

$$\begin{aligned}(\Delta \mathbf{p}_{k+1} - \mathbf{r}_{k+1}) &= Q_{k+1}^{-1} Q_k Q_k^{-1} Q_{k-1} (\Delta \mathbf{p}_{k-1} - \mathbf{r}_{k-1}) \\ &= Q_{k+1}^{-1} Q_{k-1} (\Delta \mathbf{p}_{k-1} - \mathbf{r}_{k-1})\end{aligned}$$

If we do this repeatedly we have

$$(\Delta \mathbf{p}_{k+1} - \mathbf{r}_{k+1}) = Q_{k+1}^{-1} Q_0 (\Delta \mathbf{p}_0 - \mathbf{r}_0)$$

If we let k take the integer values $0, 1, \ldots, N$ we obtain the following $6(N+1) \times 1$ dimensional vector equation

$$\begin{bmatrix} \Delta \mathbf{p}_0 - \mathbf{r}_0 \\ \hline \Delta \mathbf{p}_1 - \mathbf{r}_1 \\ \hline \Delta \mathbf{p}_2 - \mathbf{r}_2 \\ \hline \vdots \\ \hline \Delta \mathbf{p}_N - \mathbf{r}_N \end{bmatrix} = \begin{bmatrix} I \\ \hline Q_1^{-1} Q_0 \\ \hline Q_2^{-1} Q_0 \\ \hline \vdots \\ \hline Q_N^{-1} Q_0 \end{bmatrix} [\Delta \mathbf{p}_0 - \mathbf{r}_0]$$

The previous equation can be represented more concisely as

$$\mathbf{q} = U(\mathbf{q})$$

Since q is a function of $\Delta \mathbf{p}_k - \mathbf{r}_k$, $k = 0, 1, \ldots, N$ and these vectors in turn are functions of $\mathbf{p}_k$, $k = 0, 1, \ldots, N$ and $\mathbf{p}_T$, operator U may be considered as a function of $\mathbf{q}$.

Application of Contraction Operator Theory

Vector $\mathbf{q}$ can be regarded as a position vector from the origin of a $6(N+1)$ dimensional Euclidean space to the point defined by the values of its components. Thus, each

N-stage decision process can be represented by a point in this new $6(N + 1)$ dimensional space that we shall call orbit transfer space.

The matrix U can be regarded as a nonlinear operator that maps orbit transfer onto itself. It is nonlinear because the elements of matrix U depend on the coordinates of point $\mathbf{q}$. The iteration process associated with the orbit transfer process consists of making an initial estimation for point $\mathbf{q}$ denoted by $\mathbf{q}_0$, evaluating the elements of matrix U, and computing a new estimation from $\mathbf{q}_1 = U(\mathbf{q}_0)$. The successive repetition of this process can be represented by $\mathbf{q}_{n+1} = U(\mathbf{q}_n)$. If the process converges and operator U is continuous then

$$\lim_{n\to\infty} \mathbf{q}_n \to \mathbf{q}^*$$

where $\mathbf{q}^*$, the fixed point of operator U, satisfies exactly the equation

$$\mathbf{q}^* = U(\mathbf{q}^*)$$

This process will converge if U is a contraction operator. Matrix U will be a contraction operator if for any two points $\mathbf{q}'$, $\mathbf{q}''$ in orbit transfer space there exists a scalar constant $\alpha < 1$ such that[6]

$$\rho[U(\mathbf{q}'), U(\mathbf{q}'')] \leqq \alpha\rho(\mathbf{q}',\mathbf{q}'')$$

Orbit transfer space is a finite dimensional Euclidean space and the metric is defined by the norm

$$\rho[U(\mathbf{q}') - U(\mathbf{q}'')] = \|U(\mathbf{q}') - U(\mathbf{q}'')\|$$

Our convergence criterion can thus be expressed as

$$\|U(\mathbf{q}') - U(\mathbf{q}'')\|/\|\mathbf{q}' - \mathbf{q}''\| \leqq \alpha, \alpha < 1$$

The effect of various parameters, such as the number of stages in the decision process, number of iteration steps, and the size of $\|\Delta\mathbf{p}_0\|$ on convergence, can be investigated by computing the value of α.

The motion of a point mass representing a space vehicle, in terms of the time behavior of a set of two-body orbital parameters, is defined by the following vector-matrix differential equation[4,5]

$$(d\mathbf{p}/dt) = A(\mathbf{p};t)\,\mathbf{f}_w(t),\ \mathbf{p}(t_0) = \mathbf{p}_0$$

For notational convenience we shall rewrite the above differential equation in the simpler form

$$(d\mathbf{p}/dt) = \mathbf{h}(\mathbf{p};t),\ \mathbf{p}(t_0) = \mathbf{p}_0$$

where $\mathbf{h}(\mathbf{p};t)$ is a 6×1 vector whose components are functions of the two-body orbital parameters and time.[4] We know from the derivation of the differential equation that the elements of $A(\mathbf{p};t)$ are continuous functions of the two-body orbital parameters and time. Furthermore, we have bounded the values of the two-body orbital parameters, and we are considering a finite time interval $[t_0,t_0 + \tau]$. We also know from physical considerations that the components of $\mathbf{f}_w(t)$ are continuous and bounded. It then follows that the components of vector $\mathbf{h}(\mathbf{p};t)$ will be bounded, uniformly continuous functions of time defined on the closed interval $[t_0,t_0 + \tau]$ with $\mathbf{p}(t)$ restricted to a bounded region of orbital parameter space.

The approximate solution to the N-stage decision process can be represented by a set of $N + 1$ orbital parameter vectors, a set of discrete points in orbital parameter space, or a single point in orbit transfer space. It has seen shown that under suitable conditions, the iteration process produces a sequence of points in orbit transfer space that converges to the fixed point of a certain matrix operator. This fixed point represents the true solution to the N-stage decision process, and any point in the sequence represents an approximate solution. Such an approximate solution, obtained by terminating the iteration process after a finite number of steps, also represents an approximate solution to optimal orbit transfer, i.e., the previous vector differential equation. Furthermore, as the number of decisions stages and the number of iteration steps both approach infinity, the error of the approximation solution approaches zero.

This last statement follows from theorems in differential equation theory since the fixed point solution in orbital parameter space represents an Euler polygon in orbital parameter space. The vector function satisfies all the requirements for the Euler polygon to converge to the true solution of the differential equation as the number of its links i.e., the number of decision stages, approaches infinity.

In summary, the error between the approximate solution obtained by terminating the iteration process after a finite number of steps and the exact solution to the vector differential equation defining the optimal orbit transfer consists of two components. One component would vanish by letting the number of iteration steps approach infinity and the other by letting the number of decision stages approach infinity.

In actual practice there will be finite upper limits to both the number of steps in the iteration process and the number of decision stages. These limits will be set either by the desired accuracy of the approximation or by the buildup of errors inherent to the computation process.

Application to Earth-Mars Low-Thrust Orbit Transfers

To apply the computation techniques that have been discussed to the determination of optimum low-thrust orbit transfers from Earth to Mars, certain simplifying assumptions must be made. These assumptions result in an idealized representation of the real world. The practicality of this idealization depends on whether or not it produces quantitative results satisfying engineering specifications for system performance. In the absence of actual experimental results, this question can be partially answered by comparing the results with those obtained by a less idealized representation of the problem.

It will be assumed that: 1) The initial values of the state variables are the result of the motion of the space vehicle during a previous process originating either with the space vehicle on the surface of the Earth or in a geocentric parking orbit (the details of this earlier process will not be considered). 2) The perturbing forces acting on the space vehicle are caused by the gravitational attractions of the Earth, Mars, and Jupiter, and these perturbing planets move in heliocentric elliptic orbits defined by their mean or osculating orbital parameters at time t_0 and 3) The objective of the orbit transfer is to establish an artificial satellite of Mars in a circular orbit that lies in the orbital plane of Mars.

Convergence in Function Space Model for Evaluating Orbit Transfer

In the absence of actual experimental results, an evaluation of the performance of the orbit transfer process can be obtained by a less idealized representation of the problem. One way of doing this is by a more accurate integration of the equations of motion. This tends to remove some of the effects of the linearizing assumptions associated with the finite number of decision stages.

The model for evaluating the performance of the orbit transfer process consists of three separate computational processes: 1) iterative computation of a set of orbital parameter vectors representing the optimal orbit transfer process; 2) computation of the optimal control vector in terms of force per unit mass as a piecewise constant vector; and 3) integration of the nonlinear equations of motion in terms of rectangular inertial coordinates to obtain a representation of the true motion during the orbit transfer.

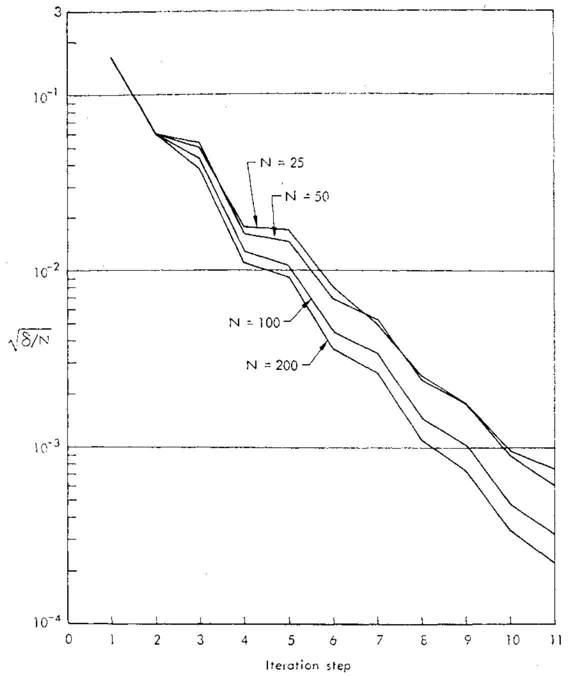

Fig. 1 Effect of the number of decision stages on convergence in orbit transfer space.

Convergence in Orbit Transfer Space and Function Space

There are two parts to the convergence problem. One part involves the convergence of the iteration process in orbit transfer space. This was discussed in terms of contraction operator theory. The second part involves the convergence of the discrete decision process to the solution of the differential equations of motion as the number of decision stages approaches infinity. In this section, the convergence of a particular orbit transfer as affected by the number of iteration steps and the number of decision stages is discussed.

The particular case chosen for the investigation has the values τ equals 216 days, λ equals unity, and a terminal satellite orbit of radius 10^{-3} a.u. Perturbations from Earth, Mars, and Jupiter are included.

The convergence data for the iteration process in orbit transfer space for 25, 50, 100, and 200 decision stages is presented in Fig. 1. The measure of convergence is the quantity $(\delta/N)^{1/2}$. The significance of $(\delta/N)^{1/2}$ is as follows. The quantity δ is the square of the norm of the difference of two $\mathbf{q}$ vectors for two successive iterations

$$\delta = \|\mathbf{q}'' - \mathbf{q}'\|^2$$

The quantity δ has the following geometric iterpretation: Let the correction for each orbital parameter Δp_i, $i = 1, \ldots, 6$, be plotted as a function of the decision stage number for two or more successive iterations. At each decision stage the ordinate difference between the curves for two successive iterations is squared, and these values are summed over the N decision stages. The quantity δ is the sum of such sums obtained for each of the six orbital parameters. It is clear that as the iteration process proceeds the curves for each parameter should converge to a limit curve. Thus, the decrease in δ with each successive iteration indicates convergence of the over-all process.

It should be noted however, that the value of δ in each case is the sum of $6N$ squared deviations. In comparing the data for different numbers of decision stages it is more meaningful to weight each δ value by $1/N$, or to compare

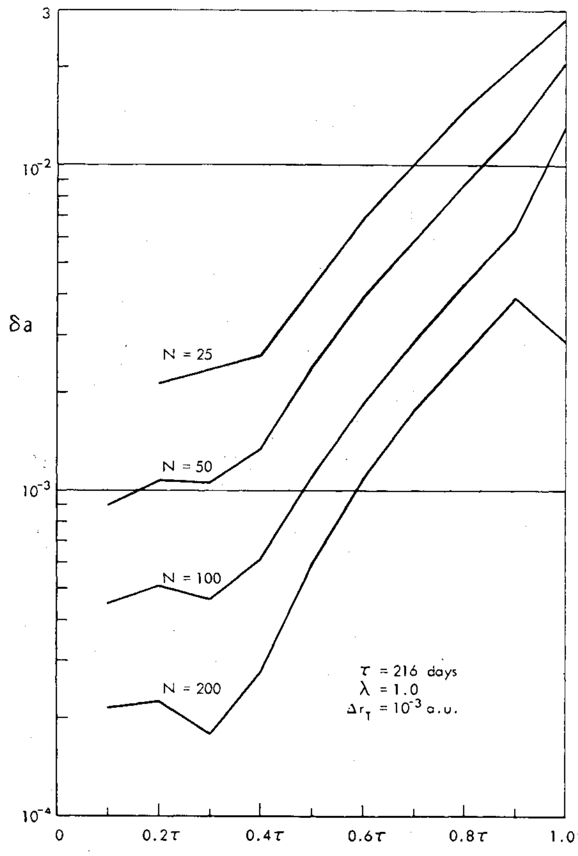

Fig. 2 Effect of the number of decision stages on the convergence of Δa in function space.

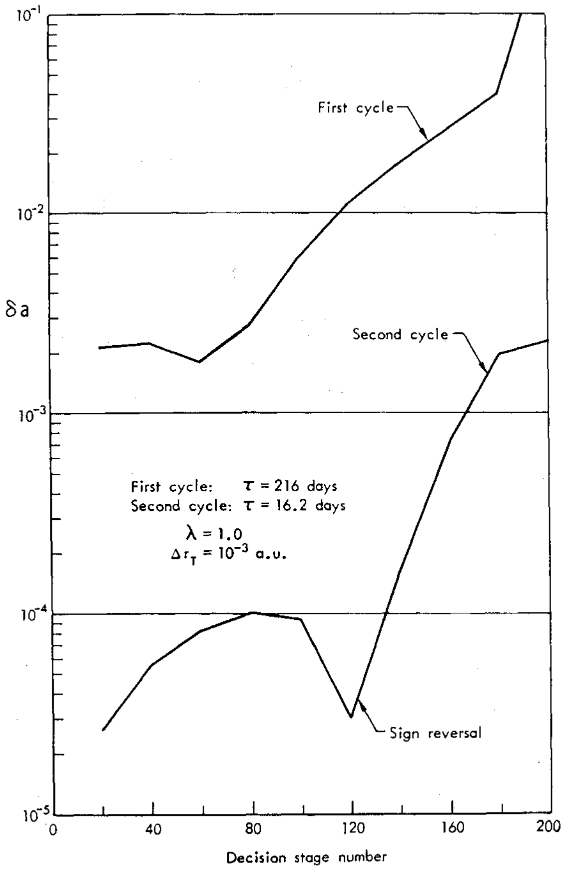

Fig. 3 Comparison of integration errors for Δa in the first and second cycle.

the parameter $(\delta/N)^{1/2}$. The convergence for all values of N is evident from the negative slopes of the curves in Fig. 1.

Several factors influence convergence in function space. It is clear from the comments earlier in this paper that increasing the number of decision stages should improve the convergence of the discrete process to the actual solution of the differential equations of motion. However, the computation of the Q matrices involves a chain of matrix multiplications, and the point will eventually be reached where the buildup of computational errors cancels any improvement caused by an increase in the number of decision stages. Furthermore, an economic limitation is present, as computation time and computer storage requirements increase with the number of decision stages.

Another factor that influences convergence in function space but is not evident from the convergence theorems is the rapid growth in magnitude of the perturbing force from Mars as the space vehicle approaches the terminal point of the orbit transfer. This factor is brought about by the necessity of computing in a heliocentric coordinate system until the vehicle is captured by the gravitational field of Mars. The magnitude of the perturbing force from Mars becomes increasingly sensitive to integration error as the space vehicle nears Mars.

The behavior of the error component Δa for the semimajor axis as the number of decision stages increases from 25 to 200 is given in Fig. 2. The buildup of the errors with time is due to the accumulation of integration error. The decrease in the errors as N increases clearly indicates convergence of the process in function space.

Initially, the computations must be carried out in a heliocentric coordinate system until the space vehicle has been captured by the gravitational field of Mars. At this time computations can be referred to a Mars-centered coordinate system and perturbations neglected, except perhaps for those caused by the figure of Mars.

Two problems exist. First, without using a very large number of decision stages (>200) the integration in the iteration process is not accurate enough to permit capture by Mars. Second, the process is an open chain process, and initial condition errors arising from previous guidance processes can result in terminal errors too great to permit capture, even with sufficiently accurate integration. Both of these problems are solved by assuming that observational data are obtained near the end of the process that permits the state of the vehicle to be computed. The iteration process is then repeated using this new initial state and a

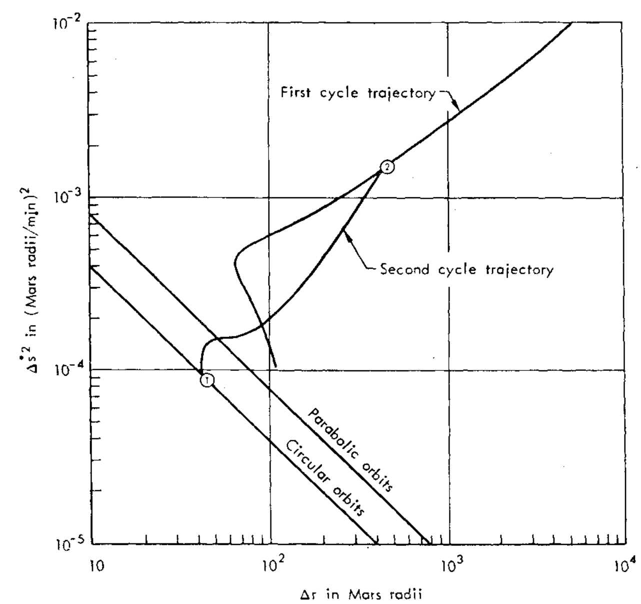

Fig. 4 Terminal phase for $\Delta r_T = 10^{-3}$ a.u.

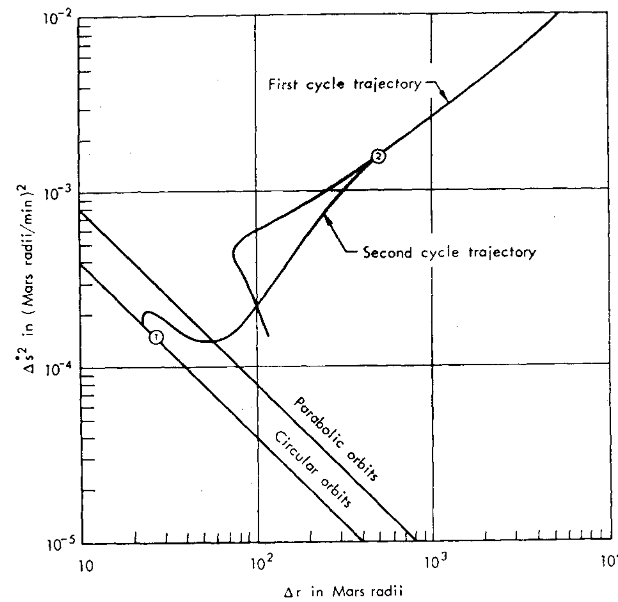

Fig. 5 Terminal phase for $\Delta r_T = 8 \times 10^{-4}$ a.u.

process duration equal to the time remaining in the original process. This procedure results in a much smaller integration error since smaller corrections in the orbital parameters are required and the time duration is shorter. The effect of a second cycle on integration errors in Δa is presented in Fig. 3. As will be shown below, capture of the space vehicle by Mars is easily achieved with the two-cycle process, provided the radius of the terminal satellite orbit is not too small.

Over-All System Performance

Once capture occurs, the resulting satellite orbit can be refined by computations made in a Mars-centered coordinate system. Since the accuracy of the computations is affected by the proximity of the terminal point to Mars, i.e., the radius of the desired satellite orbit, let us investigate this point further by selecting four circular satellite orbits with radii of 10^{-3}, 8×10^{-4}, 6×10^{-4}, and 4×10^{-4} astronomical units. These correspond to 44.13, 35.30, 26.48, and 17.65 Mars radii, respectively. Capture by Mars will occur when the position and velocity of the space vehicle relative to Mars satisfy the criterion

$$\Delta\dot{s}^2 < (2\mu/\Delta r)$$

where $\Delta\dot{s}$ is the velocity of the space vehicle relative to Mars and Δr is the distance of the space vehicle from the center of Mars. For all four cases the over-all process duration is chosen to be 216 days, and the second cycle is started 16.2 days from the terminal point. This time duration is an arbitrary choice, and in an actual orbit transfer would be determined in part by the availability of observational data. For both cycles of the four cases, λ is unity.

The curves of $\Delta\dot{s}^2$ vs Δr for satellite orbit radii of 10^{-3}, 8×10^{-4}, 6×10^{-4}, and 4×10^{-4} a.u. are shown in Figs. 4–7, respectively. The line for parabolic orbits that defines the region of capture by Mars and the line that defines the condition for circular orbits are plotted on each figure. The circle labeled 2 indicates the point on the first cycle trajectory where the second cycle starts. The circle labeled T indicates the condition for a circular satellite orbit with the desired radius. It is clear that the cases with satellite orbit radii of 10^{-3}, 8×10^{-4}, and 6×10^{-4} are captured, but the case with the terminal satellite orbit radius of 4×10^{-4} is not captured. Furthermore, the terminal position error increases as the terminal satellite radius decreases.

When the space vehicle is captured by Mars at the end of the second cycle of the interplanetary transfer, an elliptical

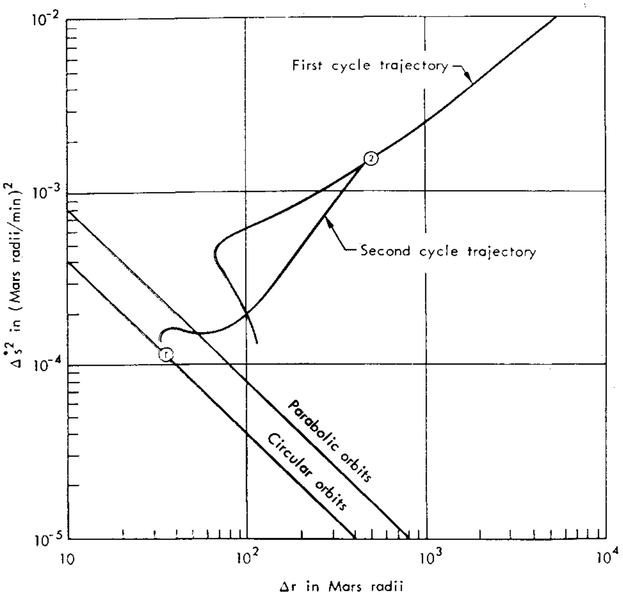

Fig. 6 Terminal phase for $\Delta r_T = 6 \times 10^{-4}$ a.u.

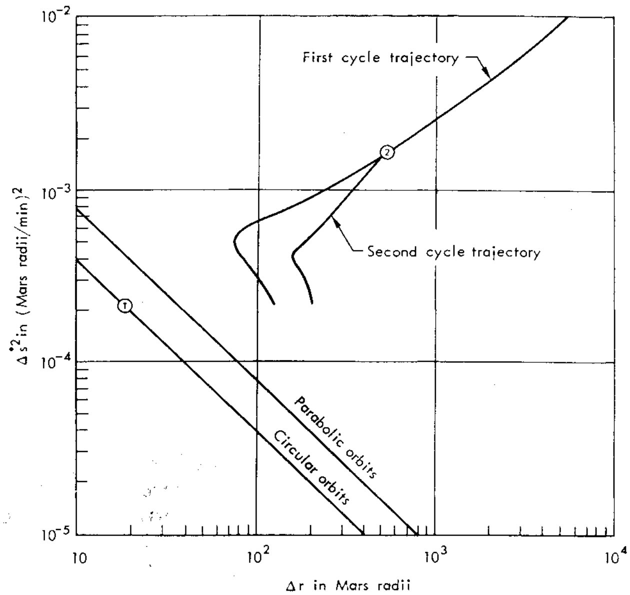

Fig. 7 Terminal phase for $\Delta r_T = 4 \times 10^{-4}$ a.u.

satellite orbit about Mars results. If this orbit is not sufficiently close to the desired terminal orbit, a sequence of observations can be taken and a third cycle used to reduce terminal errors to acceptable values. Alternatively, a new terminal orbit much closer to Mars can be selected, and the initial Martian satellite orbit corrected accordingly.

Conclusions

The functional equation of the dynamic programming solution for the interplanetary orbit transfer process has been developed, and convergence of the resultant iterative process to the optimal solution has been demonstrated through the use of contraction operator theory. Results have been obtained on the effect of the number of decision stages on convergence in orbit transfer space. In addition the effect of the number of decision stages on the convergence of Δa, the error components for the semimajor axis, are determined in function space, and a comparison of the integration errors for Δa in the first and second cycle is presented. The terminal phase behavior of the interplanetary orbital transfer process has been studied in detail for a variety of cases and capture conditions have been determined for a number of cases. A more detailed discussion of the material presented in this paper is presented in Ref. 5.

References

[1] Birkoff, G. and Rota, *Ordinary Differential Equations*, Blaisdell, Waltham, Mass., 1959.

[2] Bussard, R. W., "Some Topics in Rocket Optimization," *Optimization Techniques*, edited by G. Leitman, Academic Press, New York, 1962, Chap. 14.

[3] Smith, F. T., "Optimization of Multistage Orbit Transfer Processes by Dynamic Programming," *ARS Journal*, Vol. 31, No. 11, Nov. 1961, pp. 1553–1559.

[4] Burkhart, J. A. and Smith, F. T., "Application of Dynamic Programming to Optimizing the Orbital Control Process of a 24-Hour Communication Satellite," *AIAA Journal*, Vol. 1, No. 11, Nov. 1963, pp. 2551–2557.

[5] Smith, F. T., *The Optimization of Interplanetary Orbital Transfers by Dynamic Programming*, RM-4622-PR, July 1965, The Rand Corp.

[6] Vulikh, B. Z., *Introduction to Functional Analysis for Scientists Technologists*, (English translation), Addison-Wesley, Reading, Mass., 1962.

[7] Petrovsky, I. G., *Ordinary Differential Equations*, Prentice-Hall, Englewood Cliffs, N. J., 1966.